On the disintegration of water drops in an air stream

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A theory is developed, based on the very limited available experimental evidence, to predict the distortion and disintegration of a water drop when it is exposed to a stream of air with continuously increasing relative velocity. The theory is applied to water drops situated in the path of a solid sphere moving through the air.

1. Introduction

In the absence of aerodynamic forces, a drop of water will take up a spherical shape under the influence of surface tension. But when the drop moves through the air, aerodynamic forces are also applied to its surface and these will distort the drop from its spherical form. The distortion is of two types. For very high velocities of relative motion, such as occur near explosions, the outer surface is stripped off the drop while the central portion remains momentarily at rest. In this note, however, we shall only be concerned with the second type of distortion which only occurs at lower relative velocities.

The stages in this second type of distortion are shown in figure 1. As the relative velocity is increased from zero, the originally spherical drop (*a*), has its leading surface flattened (*b*), then the flattened surface becomes concave. The hollow increases in depth until it almost protrudes through the back of the drop. After this a spherical bubble in the shape of a bag begins to develop (*c*) and the bubble expands rapidly from the annular ring of water on which it is formed (*d*). The thin sheet of water forming the bubble eventually becomes unstable and disintegrates into tiny droplets (*e*). Soon after this the annular ring also becomes unstable and breaks up into somewhat larger droplets.

2. The experimental evidence

The experimental evidence on this subject appears to be limited to some unpublished work by Lane & Edwards (1948) in which photographs of drops in the various stages of bursting outlined above are given. This work is mainly an account of some experiments in which drops of water were dropped under gravity into a vertical wind tunnel. This apparatus consisted of a vertical tube as shown in figure 2, with a flared entry. Air was sucked in through the top and out through an exhausting fan F. Water drops D of known size were allowed to fall from a pipette down the centre line of the tube and the air velocity was adjusted until disintegration was obtained in the tube. For our purposes, the main result in this paper was a graph showing the position of a 2·38 mm diameter drop in the

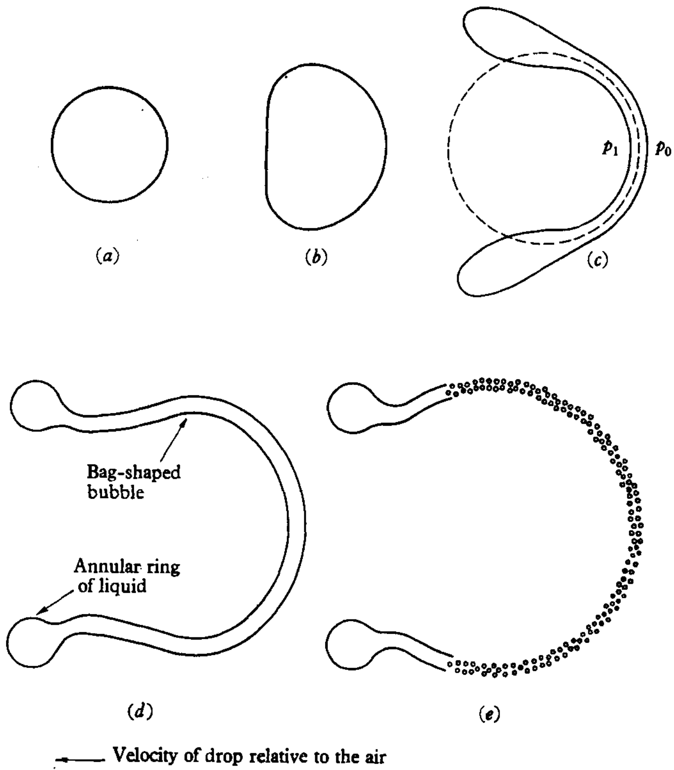
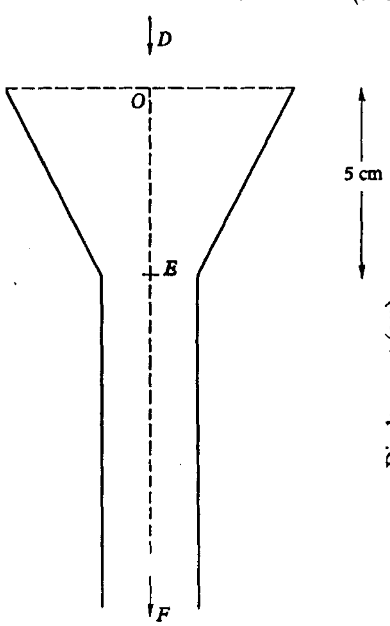


FIGURE 1 (a–e). Stages of break-up of a water drop.



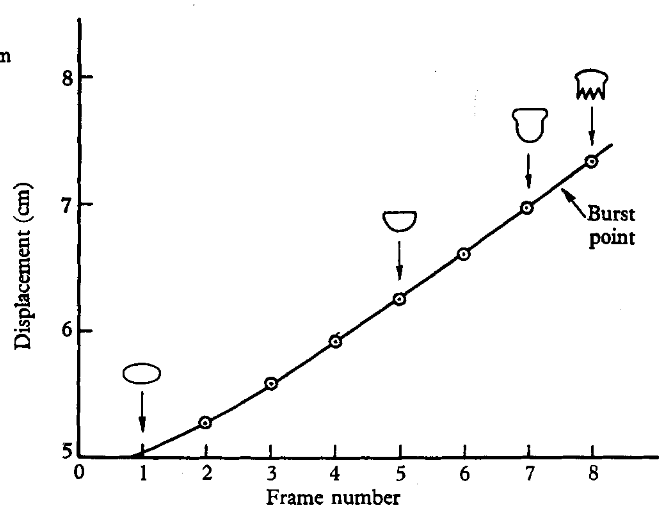


FIGURE 2 FIGURE 3

FIGURE 2. Cross-section of wind tunnel for experiment.

FIGURE 3. Motion of a water drop in the wind tunnel. Original diameter of the water drop, 2·38 mm. Air velocity, 23·2 m/sec.

tube at intervals of time of about $\frac{1}{2}$ ms. On the graph was also indicated the stage of disintegration reached by the drop at some of the time intervals. This graph is reproduced in figure 3. The displacement is the distance the drop has moved past the point O in figure 2. A good series of photographs like figure 1 was also obtained for a 2·2 mm diameter drop (see also Lane & Green 1956).

We shall also require, for our investigations, the pressure distribution on a solid sphere exposed to a stream of air. This distribution was found from experiment by Hinze (1948) to be

$$\begin{aligned} p &= \rho_a(V-u)^2\,(9\cos^2\theta-5)/8 \quad \text{for} \quad 0 \leqslant \theta \leqslant \tfrac{1}{3}\pi, \\ &= -11\rho_a(V-u)^2/32 \qquad\qquad \text{for} \quad \tfrac{1}{3}\pi \leqslant \theta \leqslant \pi, \end{aligned}$$

where ρ_a is the density of air, V is the air speed, u is the drop speed and θ is the angular distance from the point on the sphere facing the oncoming air stream. These formulae appear to be confirmed approximately by other investigations, for example, Fage (1937).

3. The condition for disintegration

The main theoretical discussion of the results of experiments on water drops has centred on ascertaining the conditions under which break-up will occur. Lane & Edwards thought that the break-up would occur roughly when the force due to the variation of aerodynamic pressure over the drop exceeds that due to surface tension. They wrote

$$4T/d = k\rho_a(V-u)^2,$$

where T is the surface tension of the liquid, d is the diameter of the drop and k is a constant. From this it follows that

$$d(V-u)^2 = \text{constant},$$

and Lane & Edwards showed from their experiments that this relation holds approximately.

Prandtl (1952, p. 325) discusses the phenomenon and arrives at a similar result. He also mentions some experiments very similar to those of Lane & Edwards which were performed by Hochschwender at Heidelberg as early as 1919. On the other hand, Hinze showed from a theoretical consideration of the break-up of freely falling drops that the condition of break-up is not an explicit equation but depends on the history of the relative velocity $(V-u)$.

Our view is that the Hinze theory is not applicable to Lane & Edwards's experiments.

To propound our theory on the matter, we must refer again to figure 1. As the relative velocity is increased, the drop takes on the form shown in (*c*) in which the bubble is just beginning to form. Let us draw a dotted sphere through this bubble. Then we contend that the critical velocity for bursting is the velocity which makes the radius of this circle a minimum. Let p_1 be the pressure just inside the bubble and p_0 the pressure just outside and let

$$p_1 - p_0 = \mu\rho_a(V-u)^2,$$

this equation defining μ which we shall regard as a constant. This equation is assumed to hold for all stages of the bubble formation. Let r be the radius of the bubble, then

$$p_1 - p_0 = 4T/r$$

and taking these equations together we get

$$r(V-u)^2 = 4T/\mu\rho_a$$

which holds for all stages of the bubble formation. Let r_m be the minimum radius of the bubble. Substituting in this equation, we obtain the following expression giving the critical velocity

$$r_m(V-u)^2_{\text{crit.}} = 4T/\mu\rho_a. \tag{1}$$

It appears from what few photographs are available that this minimum radius is about twice the radius of the original spherical drop. Hence we get essentially the same form of condition for break-up as that postulated by Lane & Edwards.

The constant μ is at this stage unknown but we can get a very rough approximation from the distribution of pressure over a solid sphere. In the distribution given above, the pressure is positive for θ from 0° to about 43° and is negative and of almost constant value over the rest of the sphere. If we take p_0 as this constant value and p_1 as the averaged pressure over the positive region of the sphere, we obtain $\mu = 0{\cdot}238 - (-0{\cdot}344) = 0{\cdot}582$.

4. The dynamics of the burst

In general the relative velocity $V-u$ is not given. Usually, V is given as a function of position s; the drop velocity u being determined by an equation of motion for the drop, which depends on V through the aerodynamic drag. The relative velocity will normally increase up to the critical velocity given by (1) after which there will be a rapid increase in the size of the drop (i.e. bubble) followed by bursting. The critical velocity conveniently divides the motion into two phases. We shall consider phase II first as it is the more interesting both theoretically and for the purpose of our applications.

For phase II we propose to idealize the problem by replacing the bubble by a hollow sphere. Let m be the mass of the original drop of water. We shall suppose a fraction f of this mass is contained in the hollow sphere and that the remaining mass is travelling along with the sphere but is not expanding. Lane & Edwards estimated that a fraction 0·3 of the mass of the original sphere is in the bubble but at present we shall not commit ourselves to this figure. Then the equations of motion for the sphere as a whole moving in a straight line are

$$\frac{ds}{dt} = u, \tag{2}$$

$$m\frac{du}{dt} = mg + \tfrac{1}{2}C_D\pi r^2\rho_a(V-u)^2, \tag{3}$$

where t is time, r is the radius of the hollow sphere and C_D is the drag coefficient. The gravity term has been put in as appropriate for comparison with the Lane & Edwards's experiments. The positive directions of s, u and V are all downwards.

To obtain the equation for the expansion of a thin spherical shell, consider a small area A on the shell. Let the thickness of the shell be δr. The mass of this area of shell is $A\,\delta r$, taking the density of water to be unity; its acceleration is d^2r/dt^2. The relation between acceleration and the force acting is thus

$$A\,\delta r \times \frac{d^2r}{dt^2} = A\left[p_1 - p_0 - \frac{4T}{r}\right] = A\left[\mu\rho_a(V-u)^2 - \frac{4T}{r}\right].$$

But the total volume of the shell is

$$4\pi r^2 \delta r = f_m, \tag{4}$$

which gives us the value of δr. Hence we obtain

$$\frac{d^2r}{dt^2} = \frac{4\pi r^2}{f_m}\left[\mu\rho_a(V-u)^2 - \frac{4T}{r}\right]. \tag{5}$$

Equations (2), (3) and (5) govern the behaviour of the drop after the critical velocity has been passed.

For phase I of the motion, the drop is idealized to a sphere of fixed radius but with a larger (but fixed) drag coefficient than that for the original spherical drop. Thus in this phase, equations (4) and (5) do not apply. This is the weakest part of the theory because the drag coefficient will obviously differ greatly for the stages (a), (b) and (c) in figure 1.

Having established the theory of the motion, out next task is to compare it with the Lane & Edwards experiment in the one case for which there was adequate information supplied. The funnel-shaped top of the wind tunnel, figure 2, was three times as wide at the top as at the bottom and its height was 5 cm. Measuring s from the point O, the air velocity is given by

$$V(s) = \frac{v}{(3-0{\cdot}4s)^2} \quad \text{for} \quad 0 \leqslant s \leqslant 5,$$
$$= v \quad \text{for} \quad 5 \leqslant s, \tag{6}$$

where v is the air velocity in the tube. The case given by Lane & Edwards was that of a drop of original diameter 0·238 cm. The graph (figure 3) shows the times at which the drop reaches a number of points down the tube. At $s = 7{\cdot}2$ cm, the radius of the bubble is about 0·47 cm and the bubble is on the point of bursting. The position $s = 7{\cdot}2$ cm is reached about 4 ms after passing $s = 5$ cm. v was measured to be 23·2 m/sec. Now from the initial gradient of the (s, t) graph, it appeared that the drop was moving at a velocity u of about 4·8 m/sec at the critical point (assumed to be pretty near $s = 5$ cm). At the critical point we also have $dr/dt = 0$ and $r = r_m = 0{\cdot}238$ cm (assuming that r_m is twice the radius of the original spherical drop). It remains to find the value of s at the critical point. This is done by finding V from equation (1) (using $u = 4{\cdot}8$ m/sec) and hence s from equation (6). From the values of the variables at the critical point, the equations were integrated on an electronic computer to find the conditions at $s = 7{\cdot}2$ cm. To do this, however, three parameters had to be chosen, namely, μ,

C_D and f. A value 0·3 was chosen for f. With this fixed, numerous integrations had to be performed with different combinations of μ and C_D in order to get an exact fit to the experimental results. It was found that μ mainly affected the rate of expansion of the bubble while C_D mainly affected the rectilinear acceleration of the drop but there was some interconnection between the two effects. A solution was eventually found with

$$f = 0{\cdot}3, \quad \mu = 0{\cdot}29, \quad C_D = 0{\cdot}5, \quad s_{\text{crit.}} = 4{\cdot}98\,\text{cm}.$$

To test the effect of changing f, a solution was also obtained with

$$f = 0{\cdot}6, \quad \mu = 0{\cdot}30, \quad C_D = 0{\cdot}5, \quad s_{\text{crit.}} = 4{\cdot}96\,\text{cm}.$$

The value of C_D is about what would be expected for a sphere and the values of μ, although somewhat smaller than that calculated from the solid sphere, seem quite plausible. So with these parameters, we felt justified in proceeding to consider other applications of the equations.

During the course of the integration trials, it was found that if μ was taken too small the bubble did not expand at all. But there was a small critical range of μ such that the bubble began to expand and then contracted again. We think that this may explain the phenomenon shown in some of Lane & Edwards's photographs in which the tip of the bubble collapses back through the annular ring.

The calculations for phase I of the motion presented less trouble as C_D is the only parameter. It was found that the motion from the pipette down to $s = 0$ was not sensibly different from a free fall under gravity, giving a velocity of $u = 2\,\text{m/sec}$. The acceleration to $4{\cdot}8\,\text{m/sec}$ required a value of about $C_D = 15$ in equation (3), using the radius of the original spherical drop for r in this equation.

5. Drops in the path of moving objects

Our main interest in the theory is to apply it to drops situated in the path of objects moving through the air to see whether these drops will be shattered or whether they will collide with the object.

The equations need no modification except (3), which need no longer contain the gravity term if the object is moving horizontally.

We consider the case of a sphere of radius a moving horizontally through the air with speed v. It is found convenient to consider the sphere to be at rest and the air to be moving. We placed the origin of co-ordinates at the point on the sphere facing the oncoming air stream and let the s-axis point in the direction in which the air was flowing. Drops are then considered as they approach the sphere along the negative s-axis.

We use parameters $f = 0{\cdot}6$, $\mu = 0{\cdot}3$ and $C_D = 0{\cdot}5$. The value chosen for f causes a slower expansion of the bubble than $f = 0{\cdot}3$ and so the probability of bursting is not overestimated. The expression for V is

$$V(s) = v - v\left(\frac{a}{a-s}\right)^3.$$

This expression is obtained from the flow of an ideal fluid round a sphere, but experimental work has shown that the expression is very nearly true for air at subsonic speeds.

Our approach to the problem consists in assuming as a first approximation that the drop travels with speed v up to the critical point. The value of s at the critical point is then given by

$$v\left(\frac{a}{a-s}\right)^3 = \left(\frac{2T}{r\mu\rho_a}\right)^{\frac{1}{2}}.$$

This is equation (1) with the appropriate values of V and u substituted and with $r_m = 2r$, where r is the radius of the originally spherical drop. From the critical point onwards the equations were integrated as previously described to see if the bubble would burst before colliding with the sphere. From the experimental evidence given by Lane & Edwards, the bursting point is taken to be the position at which the radius of the bubble is $2r_m$. But the expansion of the bubble at this point is very rapid and so the calculations are not sensitive to the choice of radius at burst. If the collision occurs before the burst then that is the end of the matter. If the burst occurs first, it is then necessary to test the validity of our assumption that the drop speed is v at the critical point. Of course, it will be slightly less than v but we must estimate by how much. From equation (3), the drag force on the drop is $F = \frac{1}{2}C_D\pi r^2\rho_a(V-u)^2$. The work done by this force up to the critical point is

$$\int_{-\infty}^{s_{\text{crit.}}} F\,ds,$$

and this is equal to the change in kinetic energy of the drop, i.e. $\frac{1}{2}m(v^2-\bar{v}^2)$ where $\bar{v}$ is the true velocity of the drop at the critical point. We can overestimate the value of the integral by putting $u = v$, in which case we get

$$1-\left(\frac{\bar{v}}{v}\right)^2 = 3\rho_a\frac{a}{r}\left[\frac{a}{a-s_{\text{crit.}}}\right]^5,$$

on using $m = \frac{4}{3}\pi r^3$ and putting $C_D = 20$ as an upper estimate. The value of $\bar{v}$ was calculated from this expression. If it differed greatly from v then the fate of the bubble became uncertain. If $\bar{v}$ turned out close to v, the bursting conclusion could be regarded as valid and a slight correction could be applied to the positions of the critical and bursting points.

We considered a range of values of r, a and v and the results are tabulated in table 1. B denotes that the drop bursts before hitting the sphere, whereas C denotes that the drop collides with the sphere. In all the B cases, $\bar{v}$ was within 5 % of v. There were two uncertain cases and these are indicated by a question mark.

It will be noted that a larger drop may collide while a smaller drop bursts for given a and v. This is in spite of the fact that a higher critical relative speed is required for the smaller drop. The explanation is that the smaller drop is more easily blown up once the critical point is past due to the smaller thickness of its bubble.

Finally, it is of interest to compare the gradients of relative velocities (at the critical points) in the above cases with the gradients in the Lane & Edwards experiments for similar drop sizes. The table below gives these gradients, the units being m/sec per cm. The values given under the heading 'sphere' are the maximum gradients for the cases we have considered in table 1 (i.e. the gradients for $a = 5$, $v = 50$):

r	Sphere	Lane & Edwards
0·2	5·33	14
0·1	8·46	20
0·05	13·43	31

It will be seen that gradients for the sphere are all less than the corresponding Lane & Edwards's gradients so we can be confident in applying the low-velocity theory in these cases.

$r = 0{\cdot}2$ cm

v	a = 5	10	20	40
200	C	C	C	B
100	C	C	C	B
50	C	C	C	B

$r = 0{\cdot}1$ cm

v	a = 5	10	20	40
200	C	C	B	B
100	C	C	B	B
50	C	C	C	?

$r = 0{\cdot}05$ cm

v	a = 5	10	20	40
200	C	B	B	B
100	C	B	B	B
50	C	C	C	?

TABLE 1. v is in m/sec, and a in cm.

6. Conclusion

In conclusion, it should be emphasized that the theory of bursting which has been put forward is not intended to be final. Indeed, considering the paucity of experimental evidence, it would be surprising if the theory does not need to be modified when further experimental results became available. It seems, however, that the basis of the theory is substantially correct and future work will need to be concentrated on determining the degree to which the 'constants' in our equations must be varied for differing drop sizes.

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